BINARY CLASSIFICATION

$$\{(2,1),\ (1,0),\ (3,1),\ (2,0)\} \sim P_{x \times y}.$$

$\mathcal{X} = \{1, 2, 3\}$ $\qquad \mathcal{Y} = \{0, 1\}.$

$h : \mathcal{X} \to \{0,1\}.$ $\qquad A = \{0, 1\}.$

$$\boxed{\{(2,1),\ (1,0),\ (3,1),\ (2,0)\}} \longrightarrow \boxed{\text{ALGO}} \longrightarrow h : \mathcal{X} \to \{0,1\}.$$

$$(1, \underline{?}) \longrightarrow h(1) = 0$$

★ WE DON'T KNOW $P_{x \times y}$ IS IN MACHINE LEARNING!

→ But if we did, we could always figure out the optimal predictor.

$$= \boxed{\mathbb{E}[\mathbb{1}\{h(1) \neq y\} \mid X=1]\, Pr(X=1)\ +}$$
$$\mathbb{E}[\mathbb{1}\{h(2) \neq y\} \mid X=2]\, Pr(X=2)\ +$$
$$\mathbb{E}[\mathbb{1}\{h(3) \neq y\} \mid X=3]\, Pr(X=3)$$

$h(1) = 1$ if $Pr(Y=1 \mid X=1) \geq 1/2.$

$\Rightarrow Pr(X=1) \cdot (\mathbb{1}\{h(1) \neq 0\} \cdot Pr(Y=0 \mid X=1) + \mathbb{1}\{h(1) \neq 1\} \cdot Pr(Y=1 \mid X=1))$

$\Rightarrow \boxed{Pr(X=1) \cdot (\underline{\mathbb{1}\{h(1)=1\}}\, Pr(Y=0 \mid X=1) + \mathbb{1}\{h(1)=0\}\, Pr(Y=1 \mid X=x))}$

1/4 $\qquad$ 3/4

1/16 $\qquad$ 3/16.

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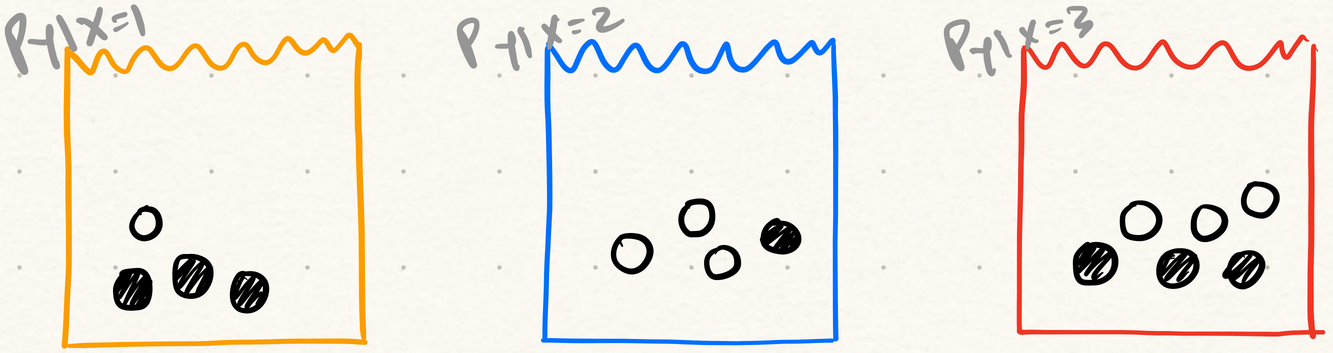
$$\boxed{\begin{array}{l} h : \{1, 2, 3\} \to \{0,1\} \\ h(1) = 1 \\ h(2) = 0 \\ h(3) = 1 \end{array}}$$



$P_{\mathcal{X}\times\mathcal{Y}}$.

$X=1$ $X=2$ $X=3$

$P(X=1) = 1/4$ $P(X=2) = 1/4$ $P(X=3) = 1/2$

$\mathcal{X} = \{1, 2, 3\}$ $\mathcal{Y} = \{0, 1\}$. (Binary classification)

X, Y are random variables on $\mathcal{X}$ and $\mathcal{Y}$.

P_{XY}

$Y \backslash X$	1	2	3
0	1/16	3/16	1/4
1	3/16	1/16	1/4

$P(X=3, Y=0) = \boxed{1/4}$.

$\mathcal{X} = \{1,2,3\}$ $\mathcal{Y} = \{0,1\}$ $\mathcal{A} = \mathcal{Y}$. (Binary classification).

$$\ell : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$$

LOSS FUNCTION: $\ell(\hat{y}, y) = \mathbb{1}\{\hat{y} \neq y\} = \begin{cases} 1 & \text{if } \hat{y} \neq y \\ 0 & \text{otherwise.} \end{cases}$

zero-one loss.

HYPOTHESIS: $h : \{1,2,3\} \to \{0,1\}$

RISK: $\boxed{R(h) = \mathbb{E}_{(x,y)\sim P_{\mathcal{X}\times\mathcal{Y}}}[\ell(h(x), y)]}$

$\ell(h(x), y)$ is a RV!

"on average, how poorly (wrt $\ell(\cdot,\cdot)$) is h doing"

Bayes Hypothesis: $h^* = \underset{h:\{1,2,3\}\to\{0,1\}}{\operatorname{argmin}} R(h)$

Bayes Risk: $R(h^*)$ (minimum value of risk).

BAYES HYPOTHESIS FOR 0-1 LOSS

$$\underset{(x,y)\sim P_{x,y}}{\mathbb{E}}[\ell(h(x),y)] = \sum_x \sum_y \ell(h(x),y)\, P(X=x, Y=y)$$

$$= \sum_{x\in\{1,2,3\}} \sum_{y\in\{0,1\}} \mathbb{1}\{h(x)\neq y\}\, P(X=x, Y=y)$$

$$= \mathbb{1}\{h(1)\neq 0\}\Pr(X=1, Y=0) + \mathbb{1}\{h(1)\neq 1\}\Pr(X=1, Y=1)$$
$$+ \dots$$

1/16

$h:\{1,2,3\}\to\{0,1\}$

$h(1) = 2.$

$h(2) = 2.$

$h(3) = 2.$

Law of Iterated Expectation:

$$\underset{(x,y)}{\mathbb{E}}[F(x,y)] = \underset{x}{\mathbb{E}}\left[\underset{y|x}{\mathbb{E}}[F(x,y)\mid x]\right]$$

$$R(h) = \underset{(x,y)}{\mathbb{E}}[\mathbb{1}\{h(x)\neq y\}]$$

$$= \underset{x}{\mathbb{E}}\left[\underset{y|x}{\mathbb{E}}[\mathbb{1}\{h(x)\neq y\}\mid x]\right]$$

$$= \mathbb{E}[\mathbb{1}\{h(1)\neq y\}\mid X=1]\Pr(X=1) +$$
$$\mathbb{E}[\mathbb{1}\{h(2)\neq y\}\mid X=2]\Pr(X=2) +$$
$$\mathbb{E}[\mathbb{1}\{h(3)\neq y\}\mid X=3]\Pr(X=3)$$

Focus on X: $\mathbb{E}_{Y|X=x}[\mathbb{1}\{h(x)\neq Y\} \mid X=x]$

$$= \mathbb{1}\{h(x)\neq 1\}\cdot \Pr(Y=1\mid X=x) + \mathbb{1}\{h(x)\neq 0\}\cdot \Pr(Y=0\mid X=x)$$

$h(x)=1$: $= \mathbb{1}\{1\neq 1\}\Pr(Y=1\mid X=x) + \mathbb{1}\{1\neq 0\}\Pr(Y=0\mid X=x)$

$= \Pr(Y=0\mid X=x)$

$h(x)=0$: $= \mathbb{1}\{0\neq 1\}\Pr(Y=1\mid X=x) + \mathbb{1}\{0\neq 0\}\Pr(Y=0\mid X=x)$

$= \Pr(Y=1\mid X=x)$

Should I choose $h(x)=1$ or $h(x)=0$?

Choose $h(x)=1$: $\Pr(Y=0\mid X=x) \leq \Pr(Y=1\mid X=x)$

$\Leftrightarrow 1-\Pr(Y=1\mid X=x) \leq \Pr(Y=1\mid X=x)$

$\Leftrightarrow \boxed{1/2 \leq \Pr(Y=1\mid X=x)}$

$$\mathbb{E}_{X}[\mathbb{E}_{Y|X}[\mathbb{1}\{h(X)\neq Y\} \mid X]]$$

$$\boxed{h^*(x) = \begin{cases} 1 & \text{if } \Pr(Y=1\mid X=x)\geq 1/2 \\ 0 & \text{otherwise} \end{cases}}$$

☆ BAYES HYPOTHESIS TYPICALLY DEPENDS ON $P_{Y|X}$.