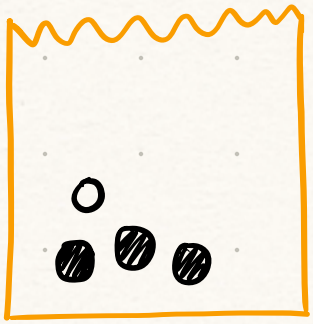
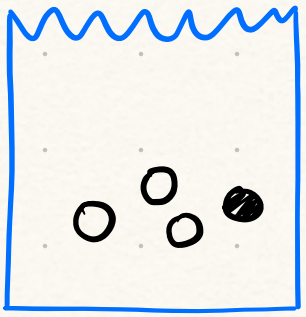
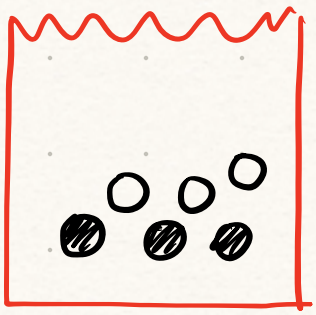


$X=1$



$X=2$



$X=3$

$P(X=1) = 1/4$ $\quad P(X=2) = 1/4$ $\quad P(X=3) = 1/2$

$x = \{1, 2, 3\}$ $\quad y = \{0, 1\}$.

X, Y are <u>random variables</u> on x and y.

y \ x	1	2	3	P_{xy}
0	$1/16$	$3/16$	$1/4$	
1	$3/16$	$1/16$	$1/4$	

$P(X=3, Y=0) = \boxed{1/4}$.

$$\boxed{E[X] = \sum_{x \in X} x \cdot P(X=x).} \qquad \left(E[X] = \int_x P_x(x)\, x\, dx\right).$$

$$= 1 \cdot 1/4 + 2 \cdot 1/4 + 3 \cdot 1/2$$

$$= 1/4 + 2/4 + 3/2 = 9/4 = \boxed{2.25}.$$

$E[Y \mid X=x]$ ← expectation of cond. dist. $P_{Y|x}$.

<u>Example</u>: $E[Y \mid X=1] = \sum_{y \in y} y \cdot P(Y \mid X=1)$

($X=2$)

$$= 0 \cdot P(Y=0 \mid X=1) + 1 \cdot P(Y=1 \mid X=1)$$

$$= P(Y=1 \mid X=1) = \boxed{3/4}$$

($X=2$)

☆ For every value of $x \in \mathcal{X}$, $P(Y|X=x)$ has a different distribution!

$P(Y|X=1)$

$P(Y=1|X=1) = 3/4$

$P(Y=0|X=1) = 1/4$

$P(Y|X=2)$

$P(Y=1|X=2) = 1/4$

$P(Y=0|X=2) = 3/4$

$P(Y|X=3)$

$P(Y=1|X=3) = 1/2$

$P(Y=1|X=3) = 1/2.$

$\mathbb{E}[Y|X=1] = 3/4$ | $\mathbb{E}[Y|X=2] = 1/4$ | $\mathbb{E}[Y|X=3] = 1/2$

Function: $F: \mathcal{X} \to \mathbb{R}$

$F(x) = \mathbb{E}[Y|X=x]$

$1 \mapsto 3/4$

$2 \mapsto 1/4$

$3 \mapsto 1/2.$

$\mathbb{E}[Y|X=x]$

- "Best guess / average" of Y given $X=x$.

☆ FUNCTIONS OF RVs are RVs!

Y is a RV then $F(Y)$ is also a Random variable.

Example: $F: \mathcal{Y} \to \mathbb{R}$

$F(0) = -2$

$F(1) = 4.$

$P(F(Y)=1) = 0.$

$P(F(Y)=-2) = P(Y=0) = 1/2$

$\vdots$

$Z = F(Y)$

$\mathbb{E}[Z] = \mathbb{E}[F(Y)] = \sum_{y \in \mathcal{Y}} f(y) \cdot Pr(Y=y)$

$\mathbb{E}[Y \mid X=x]$ functions of $x \in \mathcal{X}$.

$$\boxed{\mathbb{E}[Y \mid X] = f(X)}$$ ← Random variable.

(has a distribution... has an expectation...)

$f: \mathcal{X} \to \mathbb{R}$

$f(x) = \mathbb{E}[Y \mid X=x]$

$1 \to 3/4$

$2 \to 1/4$

$3 \to 1/2$

$\mathbb{E}_X[\mathbb{E}_Y[Y \mid X]]$ ← Just a number.

RV.

$$= \mathbb{E}_X[f(X)] = \sum_{x \in \mathcal{X}} f(x) \cdot P(X=x).$$

$$= \sum_{x \in \mathcal{X}} \mathbb{E}[Y \mid X=x] \cdot P(X=x). = \mathbb{E}[Y]. \checkmark$$

$$\boxed{\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]}$$

EXAMPLE:

$\mathbb{E}[Y \mid X=1] = 3/4$ $\quad \mathbb{E}[Y \mid X=2] = 1/4$ $\quad \mathbb{E}[Y \mid X=3] = 1/2.$

$F(x) = \mathbb{E}[Y \mid X=x]$

$1 \to 3/4$
$2 \to 1/4$
$3 \to 1/2$

} Function of x. "What do I expect Y to be w/ input x, for bag."

Instead, think of X as a RV:

$\mathbb{E}[Y \mid X]$ is a random variable

$$\mathbb{E}_X[\mathbb{E}_Y[Y \mid X]] = \sum_{x \in \mathcal{X}} \mathbb{E}[Y \mid X=x] \cdot P(X=x)$$

($f(x)$)

$$= \mathbb{E}[Y \mid X=1] \cdot P(X=1) + \mathbb{E}[Y \mid X=2] \cdot P(X=2) + \mathbb{E}[Y \mid X=3] \cdot P(X=3)$$

$$= 3/4 \cdot 1/4 + 1/4 \cdot 1/4 + 1/2 \cdot 1/2$$

$= 3/16 + 1/16 + 1/4 = 8/16 = \boxed{1/2}$

Law of Iterated Expectations: $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$

$$\mathbb{E}[Y] = \sum_{y \in \mathcal{Y}} y \cdot P(Y=y) = 0 \cdot 1/2 + 1 \cdot 1/2$$

$$= \boxed{1/2}$$

$\mathbb{E}[Y|X=x]$ — Function. $(\mathcal{X} \to \mathbb{R})$

$\mathbb{E}[Y|X]$ — Random Variable (Function of the RV X).