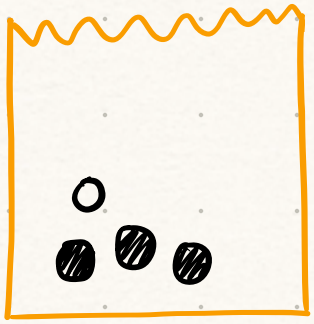
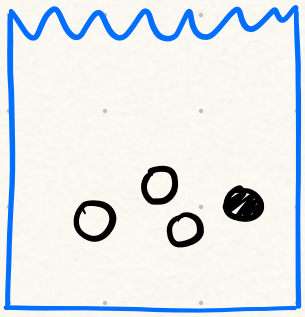


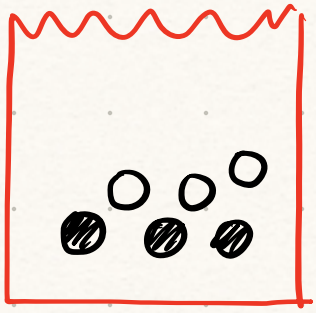
$X = 1$

$P(X=1) = 1/4$



$X = 2$

$P(X=2) = 1/4$



$X = 3$

$P(X=3) = 1/2$

$x = \{1, 2, 3\}$ $\quad y = \{0, 1\}$.

X, Y are random variables on x and y.

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

I. Marginal Distribution (P_X, P_Y)

P_X

$P(X=1) = 1/4 \quad P(X=2) = 1/4 \quad P(X=3) = 1/2$

P_Y

$P(Y=1) = 1/2 \quad P(Y=0) = 1/2.$

LAW OF TOTAL PROB / SUM RULE

$$P(Y=y) = \sum_x P(Y=y, X=x).$$

$$= \sum_x P(Y=y \mid X=x) P(X=x)$$

$$P(Y=1) = \sum_x P(Y=1 \mid X=x) \cdot P(X=x)$$

$$= P(Y=1|X=1) \cdot P(X=1) + P(Y=1|X=2) P(X=2) + P(Y=1|X=3) P(X=3)$$

$$= 3/4 \cdot 1/4 + 1/4 \cdot 1/4 + 1/2 \cdot 1/2 = \frac{3}{16} + \frac{1}{16} + \frac{1}{4} = \boxed{1/2}$$

II. Conditional Distribution ($P_{X|Y}$, $P_{Y|X}$).

☆ Conditional distributions are still probability dists.

$P_{Y|X}$:

- $P(Y=1 \mid X=1) = 3/4 \qquad P(Y=0 \mid X=1) = 1/4.$
- $P(Y=1 \mid X=2) = 1/4 \qquad P(Y=0 \mid X=2) = 3/4.$
- $P(Y=1 \mid X=3) = 1/2 \qquad P(Y=0 \mid X=3) = 1/2.$

$P_{X|Y}$: $P(X=1 \mid Y=1)$

BAYES RULE: $P(A \mid B) = \dfrac{P(B \mid A)\,P(A)}{P(B)}$ (Posterior: $P(A \mid B)$; Prior: $P(A)$)

$P(X=1) = 1/4 \xrightarrow{Y=1} P(X=1 \mid Y=1)$.

Prior → Posterior.

Bayes Rule: $P(X \mid Y) = \dfrac{P(Y \mid X)\,P(X)}{P(Y)}$.

III. Joint Distribution (P_{XY})

$P(X=x, Y=y)$.

y \ x	1	2	3	
0	1/16	3/16	1/4	$P(Y=0)=1/2$
1	3/16	1/16	1/4	$P(Y=0)=1/2$.
	$P(X=1)=1/4$	$P(X=2)=1/4$	$P(X=3)=1/2$	

$P(X,Y) = P(Y|X)P(X)$
$= P(X|Y)P(Y)$

P_{XY} is a distribution over $X \times Y$.

Gives all information about the distributions of X, Y.

$P(Y=1, X=1) = P(Y=1|X=1)P(X=1) = 3/4 \cdot 1/4$.
$P(Y=0, X=1) = P(Y=0|X=1)P(X=1) = 1/4 \cdot 1/4$
$P(Y=1, X=2) = P(Y=1|X=2)P(X=2) = 1/4 \cdot 1/4$.
$P(Y=0, X=2) = P(Y=0|X=2)P(X=2) = 3/4 \cdot 1/4$
$P(Y=1, X=3) = P(Y=1|X=3)P(X=3) = 1/2 \cdot 1/2$.
$P(Y=0, X=3) = P(Y=0|X=3)P(X=3) = 1/2 \cdot 1/2$.

$$P(Y=y|X=x) = \frac{P(Y=y, X=x)}{P(X=x)}$$

$$P(Y=1|X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{3/16}{1/4} = 12/16 = \boxed{3/4}$$

☆ ①: Product Rule. $P(Y|X)P(X) = P(X,Y)$

②: Sum Rule. $P(Y) = \sum_X P(X,Y) = \sum_X P(Y|X)P(X)$.